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C. Bernardini and C. Pellegrini: TRANSVERSE BEAM SIZE IN ELECTRON STORAGE RINGS. -

(Nota interna: n. 264)

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1. - It has been shown[1,2] that the transverse area of an electron beam in a storage ring is very sensitive to coupling fields, that is to terms in the magnetic field of the machine that can transfer radial into vertical momenta and viceversa. The reason why these effects are so important is to be found in the factor γ (the electron energy in mass units) between radial and vertical r. m. s. dimensions for the uncoupled case[3]. γ is usually so large that the uncoupled case must be considered completely unrealistic in view of the presently attained precision of magnet construction and alignements.

Moreover, because of the AdA effect[1], coupling fields can be introduced ad hoc in order to reduce loss rates; or it could prove convenient[2] to control in this way the luminosity of two crossing beams.

We want to give here general formulas to be used to extract the relevant information from the equations of motion in presence of coupling fields and radiation[4]. The generality consists in that every linear term compatible with the Maxwell equations is retained in the field[5]; thus the undamped, unforced, betatron motion can be derived from the lagrangian

$$L = \frac{1}{2}x'^2 + \frac{1}{2}z'^2 - \frac{1}{2}K^2(1-n)x^2 - \frac{1}{2}K^2 n z^2 + Mxz + + \frac{1}{2}H(xz' - z'x) - Kpx \qquad (1)$$

where x and z are radial and vertical coordinates respectively; a prime indicates derivation with respect to the arc S (to be computed on the synchronous trajectory). K(s) is the curvature of the synchronous trajectory (ST); n(s) is the field index. M is defined by

$$M = \frac{e}{\beta_s E_s}\left(\frac{\partial B_x}{\partial x}\right)_{ST}$$

where E_s and β_s are energy and velocity (c = 1) of the synchronous particle. H plays the role of a torsion and is given by

$$H = \frac{e}{\beta_s E_s} B_s$$

where B_s is the field component along the tangent to the ST. To account for closed orbit displacements there appears in **L** a term $-Kpx$ in which

$$p = \frac{E - E_s}{E_s}$$

is the percent energy deviation from the synchronous particle.

K, n, M, H are assumed to be stepwise varying functions of s; the period of the magnetic lattice will be indicated by s_o. When quadrupoles are to be described, the limits $K \to 0$, $Kn \to$ (finite quantity) must be performed.

2. - Next comes the problem of introducing radiation reaction giving rise to both damping and fluctuation forces. Detailed formulas for these forces are given in I. Here we want just note that:

a) a closed orbit can be separated out; it is in general a space curve described by equations of the form

$$x_c(s) = \xi(s)p \quad , \qquad z_c(s) = \zeta(s)p \tag{2}$$

where ξ and ζ are periodic functions of s (period s_o).

b) any correlation between betatron and synchrotron oscillations is rapidly lost.

c) the fluctuation force acting on betatron oscillations can be reduced to a sum of Dirac δ functions occuring at random times (see form. 4 below).

Thus the general structure of the equations of betatron motion is

$$y_r'' + \sum_{1,2} D_{rt} y_t' + \sum_{1,2} K_{rt} y_t = \psi_r \tag{3}$$

where $y_1 = x_b$, $y_2 = z_b$, the index b denoting the betatron part of the position coordinates. The matrices D and K are defined in I and summarize damping, coupling and ordinary focusing terms. These matrices are periodic and the same sequence repeats at intervals of length s_o.

The forcing terms ψ_r have the general structure

$$\psi_r(s) = \sum_\alpha a_r^\alpha \, \delta(s - s_\alpha) \tag{4}$$

where the coefficients a_r^α are random variables; averages $\langle \; \rangle$ will be referred to the distribution of a_r values. In particular, the only property we will use in the main calculation is

$$\langle \psi_r(s) \, \psi_t(s') \rangle = \delta(s - s') \, \phi_{rt}(s) \tag{5}$$

where $\phi_{rt}(s)$ are periodic functions of s (period s_o). The absolute (as referred to the ST) position of a particle will be given by

$$x = x_b + \xi p \qquad z = z_b + \zeta p \tag{6}$$

In the next § we want to indicate the parameters to be derived from the equations of motion in order to express the distribution in positions (x, z).

3. - Once one has determined the averages $\langle y_r y_s \rangle$, the distribution of the betatron coordinates is given by

$$P_b(y_1, y_2) = \frac{1}{\pi}\left[\det \| m_{rs} \|\right]^{1/2} \exp\left(- \sum_{1,2} m_{rs} y_r y_s\right)$$

where

$$m_{11} = \frac{1}{2} \frac{\langle y_2^2 \rangle}{\langle y_1^2 \rangle \langle y_2^2 \rangle - \langle y_1 y_2 \rangle^2}$$

$$m_{22} = \frac{1}{2} \frac{\langle y_1^2 \rangle}{\langle y_1^2 \rangle \langle y_2^2 \rangle - \langle y_1 y_2 \rangle^2}$$

$$m_{12} = m_{21} = -\frac{1}{2} \frac{\langle y_1 y_2 \rangle}{\langle y_1^2 \rangle \langle y_2^2 \rangle - \langle y_1 y_2 \rangle^2}$$

Note that

$$\det \| m_{rs} \| = \frac{1}{4}\left[\det \langle y_r y_s \rangle\right]^{-1} \qquad (r, s = 1, 2)$$

Then, the closed orbit distribution can be described by means of the variable $p = (E-E_s)/E_s$ as follows

$$P_c(p) = \frac{\alpha}{\sqrt{\pi}}(\exp - \alpha^2 p^2)$$

where

$$\alpha^2 = \frac{1}{2 \langle p^2 \rangle} .$$

The folding of P_b and P_c according to the definition (6) gives the distribution of absolute positions $P(x, z)$:

$$P(x, z) = \int_{-\infty}^{+\infty} dp \, P_b(x - \xi p, \; z - \zeta p) \, P_c(p) .$$

The explicit result is

$$P(x, z) = \frac{1}{\pi} \sqrt{\det \| \mu_{rs} \|} \; \exp\left(-\sum \mu_{rs} x_r x_s\right) \tag{7}$$

where $x_1 = x$, $x_2 = z$ and the composite parameters μ_{rs} are given by:

$$\mu_{rs} = m_{rs} - \frac{\Sigma\, m_{rv}\, m_{st}\, \eta_v\, \eta_t}{\alpha^2 + \Sigma\, m_{vt}\, \eta_v\, \eta_t} \tag{3}$$

Here $\eta_1 = \xi$, $\eta_2 = \zeta$ are the closed orbit functions. Use has been made of the symmetry property $m_{rs} = m_{sr}$; this property also holds true for the new coefficients $\mu_{rs} = \mu_{sr}$.

Local symmetry axes (somewhat like normal coordinates) can be introduced by diagonalization of the matrix $\| \mu_{rs} \|$; this can provide some information in connexion with the observation of synchrotron light (see appendix).

4. - The determination of $\langle p^2 \rangle$ introduces no new problem (compared to the uncoupled case) apart from the fact that the synchrotron damping has to be recalculated because of the more complex machine structure (see I).

Now, we want to calculate explicitly the averages $\langle y_r y_s \rangle$ for the betatron part.

Let us introduce the vectors

$$\hat{y} = \begin{vmatrix} y_1 \\ y_1' \\ y_2 \\ y_2' \end{vmatrix} = \begin{vmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \\ \hat{y}_4 \end{vmatrix}$$

$$\hat{\psi} = \begin{vmatrix} 0 \\ \psi_1 \\ 0 \\ \psi_2 \end{vmatrix} = \begin{vmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \\ \hat{\psi}_3 \\ \hat{\psi}_4 \end{vmatrix} = \begin{vmatrix} 0 \\ \hat{\psi}_2 \\ 0 \\ \hat{\psi}_4 \end{vmatrix}$$

A particular solution of (3) can be written in the form

$$\hat{y}(s) = \int_{-\infty}^{s} N(s, s')\, \hat{\psi}(s')\, ds'$$

where $N(s, s')$ is a 4×4 matrix satisfying

$$\frac{\partial}{\partial s} N(s, s') = A(s)\, N(s, s')$$

$$\frac{\partial}{\partial s'} N(s, s') = - N(s, s')\, A(s')$$

together with $N(s, s) = 1$; $A(s)$ is a 4×4 matrix directly derived from the equations of motion (3)

$$A(s) = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -K_{11} & -D_{11} & -K_{12} & -D_{12} \\ 0 & 0 & 0 & 1 \\ -K_{21} & -D_{21} & -K_{22} & -D_{22} \end{vmatrix}$$

The periodicity of the magnetic structure is better exploited by writing

$$\hat{y}(s) = N(s)\,\hat{y}(s - s_o) + \int_{s-s_o}^{s} N(s, s')\,\hat{\varphi}(s')\,ds'$$

where $N(s)$ is an abbreviation for $N(s, s-s_o)$. It proves convenient to diagonalize $N(s)$; this can be accomplished by means of a matrix $U(s)$ such that

$$\Delta(s) = U(s)\,N(s)\,U^{-1}(s)$$

is diagonal.

The transformed vector is

$$\hat{q}(s) = U(s)\,\hat{y}(s)\ .$$

Also, due to the periodicity of $N(s)$

$$\hat{q}(s - s_o) = U(s)\,\hat{y}(s - s_o)$$

so that

$$\hat{q}(s) = \Delta(s)\,\hat{q}(s - s_o) + \int_{s-s_o}^{s} U(s)\,N(s, s')\,\hat{\varphi}(s')\,ds'$$

Next introduce the correlation functions

$$Q_{rs}(s_1, s_2) = \langle \hat{q}_r(s_1)\,\hat{q}_s(s_2) \rangle$$

$$G_{rs}(s_1, s_2) = \langle \hat{q}_r(s_1)\,\hat{\varphi}_s(s_2) \rangle$$

Q and G will indicate in the following the corresponding 4x4 matrices.

Assuming that the machine is stable, a random-stationariety character of Q can be recognized in the following property

$$Q(s_1, s_2) = Q(s_1 + ks_o, s_2 + ks_o)$$

where k is an integer.

Then, recalling the property (5)[(x)]

$$\langle \hat{\varphi}_r(s)\,\hat{\varphi}_s(s') \rangle = \delta(s - s')\,\hat{\phi}_{rs}(s)$$

(x) - see pag. 6.

an equation for G is easily derived in the form

$$G(s, s') = \Delta(s)\, G(s - s_o, s') + U(s)\, N(s, s')\, \hat{\phi}(s')\, \theta(s, s')$$

Here $\theta(s, s')$ is a function (not a matrix) such that

$$\theta(s, s') = 1 \qquad \text{when } s - s_o \leq s' \leq s$$
$$= 0 \qquad \text{otherwise .}$$

It follows that, as required by causality

$$G(s, s') = 0 \qquad \text{when } s' > s$$
$$= U(s) N(s, s') \hat{\phi}(s') \qquad \text{when } s - s_o \leq s' \leq s$$

and this is all what we need, together with the random-stationariety of Q, to get the following equation for $Q(s) = Q(s, s)$

$$Q(s) = \Delta(s)\, Q(s)\, \Delta(s) + R(s) \tag{9}$$

where

$$R(s) = U(s) \int_{s-s_o}^{s} N(s, s')\, \hat{\phi}(s')\, N^T(s, s')\, ds'\, U^T(s)$$

Here U^T is the transpose of U, etc; use has been made of the obvious property $\hat{\phi} = \hat{\phi}^T$.

The solution of (9) is

$$Q(s) = \sum_{0}^{\infty} [\Delta(s)]^k\, R(s)\, [\Delta(s)]^k .$$

This series can be summed by virtue of the fact that Δ is diagonal, so that

$$Q_{rs} = \frac{1}{1 - \Delta_r \Delta_s} R_{rs} .$$

It also follows that the quantities we need are given by

$$\langle \hat{y}_r \hat{y}_s \rangle = \sum_{r', s'} \frac{U^{-1}_{rr'}\, U^{-1}_{ss'}}{1 - \Delta_{r'} \Delta_{s'}} R_{r's'} \tag{10}$$

(x) - Note of the pag. 5 :

Since $\hat{\psi}_1 = \hat{\psi}_3 = 0$, $\hat{\psi}_2 = \psi_1$, $\hat{\psi}_4 = \psi_2$ the definition of $\hat{\phi}_{rs}$ follows from (5)

$$\hat{\phi}_{11} = \hat{\phi}_{12} = \hat{\phi}_{13} = \hat{\phi}_{14} = \hat{\phi}_{23} = \hat{\phi}_{33} = \hat{\phi}_{34} = 0$$

$$\hat{\phi}_{22} = \phi_{11}; \qquad \hat{\phi}_{24} = \phi_{12}; \qquad \hat{\phi}_{44} = \phi_{22}$$

Remember that $\hat{\phi}_{rs} = \hat{\phi}_{sr}$.

This formula provides the formal solution of the problem. Its structure is better understood by writing

$$R(s) = U(s)\,\mathcal{R}(s)\,U^T(s)$$

and introducing the 4-indices simbol

$$M_{rtsv} = \sum_{r's'} \frac{U^{-1}_{rr'}\,U_{r't}\,U^{-1}_{ss'}\,U_{s'v}}{1 - \Delta_{r'}\,\Delta_{s'}}$$

Then $\langle \hat{y}_r \hat{y}_s \rangle$ can be expressed as the product of a factor depending on machine structure only (M) and a factor depending on radiation ($\mathcal{R}$):

$$\langle \hat{y}_r \hat{y}_s \rangle = \sum_{t,v} M_{rtsv}\,\mathcal{R}_{tv}$$

In particular, resonant filtering of the radiation noise is exhibited in the denominators $1 - \Delta_{r'}\,\Delta_{s'}$ appearing in the definition of M_{rtsv}; analysis of such denominators for a given machine structure is generally important in itself. A practical case will be shown in the next §.

5. - In order to see how form.(10) works, a simple example will now be calculated somewhat further. The example consists in studying a strong focusing machine in which a localized coupling has been inserted (a very short rotated quadrupole lens). Numerical computations will not be performed since the variety of existing or designed machines would not be comparable with a too special case. It should be noted that the algebra involved is relatively simple until the closed orbit functions, the damping constants, the various matrices U(s), N(s, s') are not to be computed in detail; at that point a computer is necessarily needed.

Coming back to the example we want to examine, let us introduce at a given point in the machine (otherwise uncoupled) a 45° rotated quadrupole of zero length. This can be described by putting in eq. (3) for the coupling terms

$$D_{12} = D_{21} = 0$$

$$K_{12} = k_{12} \sum_n \delta(s - n\,s_o)$$

$$K_{21} = k_{21} \sum_n \delta(s - n\,s_o)$$

Because of Maxwell equations, it must also be that $k_{12} = k_{21} = -m$ where the definition of m is given by(x) (compare the lagrangian (1))

$$M(s) = m \sum \delta\,(s - n\,s_o)$$

(x) - see pag. 8.

The corresponding transfer matrix over the quadrupole is

$$\mathcal{C} = \begin{vmatrix} I & +C \\ +C & I \end{vmatrix}$$

where I is the 2 x 2 unit matrix and

$$C = \begin{vmatrix} 0 & 0 \\ m & 0 \end{vmatrix}$$

Note that $\det \mathcal{C} = 1$.

Assuming that there is a coupling element at the end of each period, the matrix N(s) is of the form

$$N(s) = \mathcal{C}\,\mathcal{F}(s) = \begin{vmatrix} I & +C \\ +C & I \end{vmatrix} \begin{vmatrix} F_x & 0 \\ 0 & F_z \end{vmatrix} = \begin{vmatrix} F_x & +CF_z \\ +CF_x & F_z \end{vmatrix}$$

Since det F = 1 neglecting dampings, det N is also equal to 1 to the same approximation.

The main task to be accomplished is now to get the eigenvalues of N(s), that is to solve

$$\det (N - \lambda I) = 0 .$$

This equation is equivalent to

$$\lambda^4 - \lambda^3 \, Tr(F_x + F_z) + \lambda^2 \left\{ \det F_x + \det F_z + Tr F_x \, Tr F_z - \right.$$

$$\left. - m^2 X_{12} Z_{12} \right\} - \lambda \left\{ Tr F_x \det F_z + Tr F_z \det F_x \right\} +$$

$$+ \det F_x \det F_z = 0$$

where

$$X_{12} = (F_x)_{12} , \qquad Z_{12} = (F_z)_{12}$$

(x) - Note of the pag. 7:

According to the definition of M, m is also given by

$$m = \frac{el}{\beta_s E_s} \left(\frac{\partial B_x}{\partial x} \right)_{ST}$$

where l is the effective length of the quadrupole. A more practical formula is obtained by introducing the radius of curvature R of an electron of momentum $\beta_s E_s$ in a given reference field B:

$$m = \frac{1}{R} \, \frac{1}{B} \left(\frac{\partial B_x}{\partial x} \right)_{ST} .$$

and, when the dampings can be neglected,

$$\det F_x = \det F_z = 1 .$$

Put $\mathrm{Tr}\, F_{x,z} = 2 \cos \mu_{x,z}$ and $X_{12} = \beta_x \sin \mu_x$, $Z_{12} = \beta_z \sin \mu_z$ to recall the usual parameters of the uncoupled machine (m = 0). Then, neglecting dampings

$$\lambda^4 - 2(\cos\mu_x + \cos\mu_z)\,\lambda^3 + \lambda^2 (2 + 4\cos\mu_x \cos\mu_z - m^2 \beta_x \beta_z \cdot$$

$$\cdot \sin\mu_x \sin\mu_z) - 2\lambda(\cos\mu_x + \cos\mu_z) + 1 = 0 .$$

This can be also written

$$(\lambda + \frac{1}{\lambda})^2 - 2(\cos\mu_x + \cos\mu_z)(\lambda + \frac{1}{\lambda}) + 4\cos\mu_x \cos\mu_z -$$

$$- m^2 \beta_x \beta_z \sin\mu_x \sin\mu_z = 0$$

Solutions are

$$\lambda + \frac{1}{\lambda} = \cos\mu_x + \cos\mu_z \pm \sqrt{(\cos\mu_x - \cos\mu_z)^2 + m^2 \beta_x \beta_z \sin\mu_x \sin\mu_z}$$

$$= a \pm b .$$

Put $\lambda = e^{ig}$ and $\cos g_\pm = \frac{1}{2}(a \pm b)$, then the conditions for g to be real (stability) are

$$|a \pm b| \leq 2$$

$$(\cos\mu_x - \cos\mu_z)^2 + m^2 \beta_x \beta_z \sin\mu_x \sin\mu_z > 0$$

In the stable case (with no dampings) the eigenvalues λ are $\exp(\pm ig_+)$ and $\exp(\pm ig_-)$: they coincide with the elements of the diagonal matrix Δ apart from the dampings. We choose in the following

$$\Delta_1 = \exp(+ig_+) \qquad \Delta_3 = \exp(+ig_-) \qquad (\Delta_{ii} = \Delta_i)$$

$$\Delta_2 = \exp(-ig_+) \qquad \Delta_4 = \exp(-ig_-)$$

in the limit of zero damping.

Next introduce the dampings by noting that

$$\det F_x \cong 1 - D_{11} s_o , \qquad \det F_z \cong 1 - D_{22} s_o$$

$$\mathrm{Tr}\, F_x \simeq 2\cos\mu_x (1 - \frac{1}{2} D_{11} s_o)$$

$$\mathrm{Tr}\, F_z \simeq 2\cos\mu_z (1 - \frac{1}{2} D_{22} s_o)$$

$$X_{12} = \beta_x \sin\mu_x \left(1 - \frac{1}{2} D_{11} s_o\right)$$

$$Z_{12} = \beta_z \sin\mu_z \left(1 - \frac{1}{2} D_{22} s_o\right)$$

After some algebra, the elements of Δ including damping to the first order can be put in the form

$$\Delta_{1,2} = \exp(\pm i g_+ - \omega_+)$$

$$\Delta_{3,4} = \exp(\pm i g_- - \omega_-)$$

where $\omega_\pm$ are complicate functions of $g_\pm$, μ_x, μ_z, β_x, β_z, m^2, D_{11}, D_{22}. Note that to the first order in $D_{ii} s_o$, $\omega_\pm$ is in general complex indicating a frequency shift. In any practical case $\mathrm{Im}\,\omega_\pm \ll g_\pm$ so that only $\mathrm{Re}\,\omega_\pm$ will contribute significantly. In the following $\omega_\pm$ is written in place of $\mathrm{Re}\,\omega_\pm$.

It is now seen that when $g_+ \pm g_- \neq 2n\pi$ and $g_\pm \neq n\pi$ (n an integer) the only relevant terms in (10) are those associated with

$$1 - \Delta_1 \Delta_2 \simeq 2\,\omega_+$$

$$1 - \Delta_3 \Delta_4 \simeq 2\,\omega_- .$$

Thus, in general, the following formula holds far from resonances

$$M_{rtsv} \simeq \frac{1}{2\omega_+} U^{-1}_{r1} U_{1t} U^{-1}_{s2} U_{2v} + \frac{1}{2\omega_-} U^{-1}_{r3} U_{3t} U^{-1}_{s4} U_{4v} .$$

At this point the matrix U has to be computed; we shall not solve this lenghty but trivial problem explicitly.

It must be noted that in the calculation of $\mathcal{R}$ only 4 non-zero elements of the matrix $\hat{\varphi}$ occur because of the definition of the vector $\hat{\psi}$. A lenghty calculation is however to be performed in actually deriving the matrix N(s, s').

The problem is further reduced by the fact that we need only the quantities $\langle \hat{y}_1^2 \rangle = \langle x_b^2 \rangle$, $\langle \hat{y}_3^2 \rangle = \langle z_b^2 \rangle$, $\langle \hat{y}_1 \hat{y}_3 \rangle = \langle x_b z_b \rangle$.

It also happens that only one component of the random force is usually large (the one along the curvature radius).

APPENDIX. -

The distribution in positions (7) can be expressed in an equivalent form in which local symmetry axes appear. To this end, let us introduce locally rotated axes $\bar{x}$, $\bar{z}$ by the transformation

$$\bar{x} = x \cos \theta + z \sin \theta$$

$$\bar{z} = - x \sin \theta + z \cos \theta \ .$$

By choosing

$$\mathrm{tg}\, 2\theta = \frac{2\mu_{xz}}{\mu_{xx} - \mu_{zz}}$$

and then putting

$$\bar{\mu}_{xx} = \frac{1}{2}(\mu_{xx} + \mu_{zz}) + \frac{\mu_{xz}}{\sin 2\theta}$$

$$\bar{\mu}_{zz} = \frac{1}{2}(\mu_{xx} + \mu_{zz}) - \frac{\mu_{xz}}{\sin 2\theta}$$

the distribution transforms into

$$P(\bar{x}, \bar{z}) = \frac{1}{\pi}(\bar{\mu}_{xx}\bar{\mu}_{zz})^{1/2} \exp - (\bar{\mu}_{xx}\bar{x}^2 + \bar{\mu}_{zz}\bar{z}^2) \ .$$

Thus, looking at the beam section (e. g. by the light) an elliptic spot appears having symmetry axes along the directions $\bar{x}$ and $\bar{z}$. Also, the inverse effective area S in the luminosity (for two crossing overlapping beams) is given by

$$\frac{1}{S} = \int P^2(\bar{x}, \bar{z})\, d\bar{x}\, d\bar{z} = \frac{1}{2\pi}(\bar{\mu}_{xx}\bar{\mu}_{zz})^{1/2} = \frac{1}{2} P(0,0) \ .$$

$1/S$ is thus half the maximum density.

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(3) - C. Pellegrini, Nuovo Cimento, suppl. 22, 603 (1961).

(4) - G. Leleux and P. Gratreau have kindly send to us unpublished work concerning the same problem (Orsay, Internal reports 14-64 and 22-64).

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